

# Approximation via regularization of the local time of semimartingales and Brownian motion

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## Abstract

Through a regularization procedure, a few schemes for approximation of the local time of a large class of continuous semimartingales and reversible diffusions are given. The convergence holds in the ucp sense. In the case of standard Brownian motion, we have been able to bound the rate of convergence in  $L^2$ , and to establish the a.s. convergence of some of our schemes.

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## 1. Introduction

We consider a complete probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  which satisfies the usual hypotheses. In our study,  $X$  will stand for a continuous  $(\mathcal{F}_t)$ -semimartingale. Let  $X_t = M_t + V_t$  be its canonical decomposition, where  $M$  is a local martingale and  $V$  is an adapted process with finite variation.  $(L_t^x(X), 0 \leq t, x \in \mathbb{R})$  will denote the local time process associated with  $X$  and  $\langle X \rangle$  stands for the quadratic variation of  $X$ .

1. Let  $\epsilon > 0$ . Let us introduce a family  $(J_\epsilon(t, y), y \in \mathbb{R}, t \geq 0)$  of processes which will play a central role in our study:

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$$J_\epsilon(t, y) = \frac{1}{\epsilon} \int_0^t (\mathbb{1}_{\{y < X_{s+\epsilon}\}} - \mathbb{1}_{\{y < X_s\}}) (X_{s+\epsilon} - X_s) ds. \quad (1.1)$$

It is actually possible to prove that the measures  $(J_\epsilon(t, y)dy)$  on  $\mathbb{R}$  weakly converge to  $(L_t^y(X)dy)$  as  $\epsilon \rightarrow 0$  (see Section 1.1.3 of [2] for the proof). Namely, for any continuous function  $f$  with compact support,

$$\lim_{\epsilon \rightarrow 0} (\text{ucp}) \int_{\mathbb{R}} f(y) J_\epsilon(t, y) dy = \int_0^t f(X_s) d\langle X \rangle_s = \int_{\mathbb{R}} f(y) L_t^y(X) dy.$$

As a result, it seems natural to study the convergence of  $J_\epsilon(t, y)$  to  $L_t^y(X)$  when  $\epsilon \rightarrow 0$  and  $y$  is a fixed real number.

2. Our first approximation result concerns  $(J_\epsilon(t, y))$  when  $X$  belongs to a class of diffusions stable under time reversal. This kind of diffusion has been studied in [7,6,14]. We consider the generalization made in Section 5 of [14]. We name here a *reversible diffusion* a process  $X$  which satisfies  $X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds$  and the conditions of Theorem 5.1 of [14].

With a fixed  $T > 0$ , we associate the process

$$\tilde{X}_u = X_{T-u}, \quad u \in [0, T]. \quad (1.2)$$

If  $X$  is a reversible diffusion, according to Theorem 5.1 of [14], then  $(\tilde{X}_u)_{u \in [0, T]}$  is a diffusion and  $d\langle \tilde{X} \rangle_s = \sigma(T-s, \tilde{X}_s)^2 ds$ .

**Theorem 1.1.** *Let  $X$  be a reversible diffusion. Then*

$$\lim_{\epsilon \rightarrow 0} (\text{ucp}) J_\epsilon(t, y) = L_t^y(X), \quad \forall y \in \mathbb{R}.$$

Our limit in Theorem 1.1 is valid when  $y$  is fixed. We have not been able to prove that the convergence is uniform with respect to  $y$  varying in a compact set. For simplicity of notation, we take  $y = 0$  and we write  $J_\epsilon(t)$  instead of  $J_\epsilon(t, 0)$  in the sequel of the paper.

3. The proof of Theorem 1.1 is based on a decomposition of  $J_\epsilon(t)$  as a sum of two terms. Let us briefly present this decomposition. Using

$$\mathbb{1}_{\{X_{(u+\epsilon) \wedge t} > 0\}} - \mathbb{1}_{\{X_u > 0\}} = \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} > 0, X_u \leq 0\}} - \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} \leq 0, X_u > 0\}}, \quad (1.3)$$

and developing the product gives  $J_\epsilon(t) = I_\epsilon^3(t) + I_\epsilon^4(t) + R_\epsilon(t)$ , where

$$I_\epsilon^3(t) = \frac{1}{\epsilon} \int_0^t X_{(u+\epsilon) \wedge t}^+ \mathbb{1}_{\{X_u \leq 0\}} du + \frac{1}{\epsilon} \int_0^t X_{(u+\epsilon) \wedge t}^- \mathbb{1}_{\{X_u > 0\}} du, \quad (1.4)$$

$$I_\epsilon^4(t) = \frac{1}{\epsilon} \int_0^t X_u^- \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} > 0\}} du + \frac{1}{\epsilon} \int_0^t X_u^+ \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} \leq 0\}} du, \quad (1.5)$$

and  $(R_\epsilon(t))_{t \geq 0}$  is a process which goes to 0 a.s. as  $\epsilon \rightarrow 0$ , uniformly on compact sets in time (see point 3. of Section 2 for details). Note that in (1.3) and (1.4) and (1.5) we have systematically introduced  $X_{(u+\epsilon) \wedge t}$  instead of  $X_{u+\epsilon}$  in order to guarantee that  $I_\epsilon^3(t)$ ,  $I_\epsilon^4(t)$  are adapted processes.

**Theorem 1.2.** (i) *If  $X$  is a continuous semimartingale, then*

$$\lim_{\epsilon \rightarrow 0} (\text{ucp}) I_\epsilon^3(t) = \frac{1}{2} L_t^0(X).$$

(ii) If  $X$  is a reversible diffusion, then

$$\lim_{\epsilon \rightarrow 0} (\text{ucp}) I_{\epsilon}^4(t) = \frac{1}{2} L_t^0(X).$$

**Remark 1.3.** 1. It is clear that [Theorem 1.2](#) implies that [Theorem 1.1](#) holds.

2. We would like to emphasize that the choice of strict or large inequalities in [\(1.4\)](#) and [\(1.5\)](#) is important. Indeed, the Lebesgue measure of  $\{u; X_u = 0\}$  may not vanish. Note that, in Tanaka's formula,  $X_t^+$  is associated with  $\mathbb{1}_{\{X_t > 0\}}$ . This heuristically explains our choice.

5. In the Brownian case, we can go further than [Theorems 1.1](#) and [1.2](#). Let us explain our extension which concerns [Theorem 1.2](#). Obviously,  $I_{\epsilon}^3(t)$  and  $I_{\epsilon}^4(t)$  can be decomposed as

$$I_{\epsilon}^3(t) = I_{\epsilon}^{3,1}(t) + I_{\epsilon}^{3,2}(t) + r_{\epsilon}^3(t), \quad I_{\epsilon}^4(t) = I_{\epsilon}^{4,1}(t) + I_{\epsilon}^{4,2}(t) + r_{\epsilon}^4(t),$$

where

$$I_{\epsilon}^{3,1}(t) = \frac{1}{\epsilon} \int_0^t X_{(u+\epsilon) \wedge t}^- \mathbb{1}_{\{X_u > 0\}} du, \quad I_{\epsilon}^{3,2}(t) = \frac{1}{\epsilon} \int_0^t X_{(u+\epsilon) \wedge t}^+ \mathbb{1}_{\{X_u < 0\}} du, \quad (1.6)$$

$$I_{\epsilon}^{4,1}(t) = \frac{1}{\epsilon} \int_0^t X_u^- \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} > 0\}} du, \quad I_{\epsilon}^{4,2}(t) = \frac{1}{\epsilon} \int_0^t X_u^+ \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} < 0\}} du, \quad (1.7)$$

$$r_{\epsilon}^4(t) = \frac{1}{\epsilon} \int_0^t X_u^+ \mathbb{1}_{\{X_{(u+\epsilon) \wedge t} = 0\}} du, \quad r_{\epsilon}^3(t) = \frac{1}{\epsilon} \int_0^t X_{(u+\epsilon) \wedge t}^+ \mathbb{1}_{\{X_u = 0\}} du. \quad (1.8)$$

**Theorem 1.4.** Let  $X$  be a standard Brownian motion. Then,

$$(1) \lim_{\epsilon \rightarrow 0} (\text{ucp}) I_{\epsilon}^{3,1}(t) = \lim_{\epsilon \rightarrow 0} (\text{ucp}) I_{\epsilon}^{3,2}(t) = \frac{1}{4} L_t^0(X).$$

$$(2) \lim_{\epsilon \rightarrow 0} (\text{ucp}) I_{\epsilon}^{4,1}(t) = \lim_{\epsilon \rightarrow 0} (\text{ucp}) I_{\epsilon}^{4,2}(t) = \frac{1}{4} L_t^0(X).$$

Moreover, for all  $T > 0$  and  $\delta \in ]0, \frac{1}{2}[$ , there exists a constant  $C$  such that

$$\left\| \sup_{t \in [0, T]} \left| I_{\epsilon}^{4,i}(t) - \frac{1}{4} L_t^0(X) \right| \right\|_{L^2(\Omega)} \leq C \epsilon^{\frac{\delta}{2}}, \quad i = 1 \text{ or } 2. \quad (1.9)$$

Then, let us give below complements related to [Theorem 1.1](#).

**Theorem 1.5.** Let  $X$  be the standard Brownian motion. For all  $T > 0$ ,  $x \in \mathbb{R}$  and  $\delta \in ]0, \frac{1}{2}[$ , there exists a constant  $C$  such that

$$\forall \epsilon \in ]0, 1], \quad \left\| \sup_{t \in [0, T]} \left| J_{\epsilon}(t) - L_t^0(X) \right| \right\|_{L^2(\Omega)} \leq C \epsilon^{\frac{\delta}{2}}. \quad (1.10)$$

In the setting of stochastic integration by regularization (see for instance [\[11,9,10,12,13\]](#)), the ucp convergence is mainly used. So, [Theorems 1.1](#), [1.2](#) and [1.4](#) are of this type. There exist a few results using almost sure convergence in [\[4\]](#). Our version of [Theorem 1.1](#) formulated in terms of a.s. convergence is the following.

**Proposition 1.6.** *Let  $X$  be the standard Brownian motion and  $(\epsilon_n)_{n \in \mathbb{N}}$  be a decreasing sequence of non-negative real numbers which satisfies  $\sum_{i=1}^{\infty} \sqrt{\epsilon_i} < \infty$ . Then, for any  $x \in \mathbb{R}$ ,  $i = 1, 2$ , almost surely,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| I_{\epsilon_n}^{4,i}(t, x) - \frac{1}{4} L_t^x(X) \right| = 0, \quad \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |J_{\epsilon_n}(t, x) - L_t^x(X)| = 0.$$

6. Many approximation schemes of the local time of semimartingale already exist. First, we deduce easily from the occupation times formula and right continuity of  $x \rightarrow L_t^x(X)$  that  $\frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{x \leq X_s \leq x+\epsilon\}} d\langle X \rangle_s$  tends to  $L_t^x(X)$  as  $\epsilon \rightarrow 0$ . The Lévy excursion theory (cf. [15]) gives other approximations via the count of the downcrossings number or of the excursion number before a given time. For the case of diffusion, the convergence of normalized sums to local time has been studied in [1,5].

7. Let us briefly detail the organization of the paper. Section 2 contains conventions and preliminary lemmas. The proof of Theorem 1.2 is given in Section 3. The Brownian case is studied in Section 4 (proof of Theorem 1.4) and in Section 5 (proofs of Theorem 1.5 and Proposition 1.6).

Some results of this paper were announced without any proof in [3]. Moreover, the setting in [3] was the Brownian one.

## 2. Preliminary lemmas and conventions

1. Let us start with some conventions, which will be used in the sequel of the paper:

- $[0, T]$  will denote a given compact interval of time.
- In the calculations,  $C$  will denote a generic constant (random or not). If  $C$  is random, then  $C \in L^2(\Omega)$ .
- We fix  $\delta \in ]0, \frac{1}{2}[$ . Then, the Brownian motion (resp. its local time) is Hölder continuous of order  $\delta$  in time (resp. in space).
- $\Phi$  will be the distribution function of the standard Gaussian distribution.
- $R_\epsilon(t)$  is the generic notation for a process which converges to 0 as  $\epsilon \rightarrow 0$ , uniformly on the compact set in time (see point 3.).

2. The two following Lemmas 2.1 and 2.2 will be useful tools for proving many of our results.

**Lemma 2.1.** *Let  $(Y_t)_{t \geq 0}$  be a continuous martingale and  $(H_\epsilon(s, u), s \in [0, T], u \in [0, T], \epsilon \in [0, 1])$  be a collection of predictable processes which are measurable with respect to  $(s, u, \omega)$ . We suppose that*

$$\lim_{\epsilon \rightarrow 0} E \left[ \int_0^T \left( \int_{(s-\epsilon)^+}^s H_\epsilon(s, u) du \right)^2 d\langle Y \rangle_s \right] = 0, \quad (2.1)$$

and that one of the two following conditions holds:

- (1)  $(H_\epsilon(s, u))_{\epsilon \in ]0, 1], s, u \in [0, T]}$  is uniformly bounded and  $\langle Y \rangle_T$  is bounded.
- (2)  $Y$  is the standard Brownian motion and  $\int_0^T \int_{(s-\epsilon)^+}^s H_\epsilon^2(s, u) du ds < \infty$ .

Then  $\sup_{t \in [0, T]} \left| \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} H_\epsilon(s, u) dY_s \right) du \right|$  converges to 0 in  $L^2(\Omega)$  as  $\epsilon \rightarrow 0$ .

Moreover, in the case of conditions (2), if there exists  $\alpha > 0$  such that  $E[\int_0^T (\int_{(s-\epsilon)^+}^s H_\epsilon(s, u) du)^2 ds] \leq C\epsilon^\alpha$ , then

$$E \left( \sup_{0 \leq t \leq T} \left[ \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} H_\epsilon(s, u) dY_s \right) du \right]^2 \right) \leq C\epsilon^\alpha.$$

**Proof.** Let us study  $K_\epsilon(t) := \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} H_\epsilon(s, u) dY_s \right) du$ .

In the case of conditions (1) or (2), the stochastic Fubini theorem (see Section IV.5 of [8,2] for details) applies and we get

$$K_\epsilon(t) = \int_0^t \left( \int_{(s-\epsilon)^+}^s H_\epsilon(s, u) du \right) dY_s.$$

Then, (2.1) implies that  $E(\langle K_\epsilon \rangle_t) < \infty$ . Consequently,  $(K_\epsilon(t))_{t \in [0, T]}$  is a martingale which is bounded in  $L^2(\Omega)$ .

Then, we are allowed to apply Doob's inequality in both cases:

$$E \left( \sup_{0 \leq t \leq T} (K_\epsilon(t))^2 \right) \leq 4E \left[ \int_0^T \left( \int_{(s-\epsilon)^+}^s H_\epsilon(s, u) du \right)^2 d\langle Y \rangle_s \right].$$

Condition (2.1) ends the proof.  $\square$

A version of the following lemma without the rate of convergence can be found in Section XIII. 2 of [8].

**Lemma 2.2.** Let  $X$  be the standard Brownian motion and  $f$  a Borel function such that  $f$  and  $x \rightarrow x^\delta f(x)$  are integrable. Then,

$$\left| \frac{1}{\sqrt{\epsilon}} \int_0^t f \left( \frac{X_u}{\sqrt{\epsilon}} \right) du - \left( \int_{\mathbb{R}} f(y) dy \right) L_t^0(X) \right| \leq C\epsilon^{\frac{\delta}{2}}, \quad \forall t \in [0, T].$$

**Proof.** According to the occupation times formula, we have

$$\frac{1}{\sqrt{\epsilon}} \int_0^t f \left( \frac{X_u}{\sqrt{\epsilon}} \right) du = \frac{1}{\sqrt{\epsilon}} \int_{\mathbb{R}} f \left( \frac{x}{\sqrt{\epsilon}} \right) L_t^x(X) dx.$$

The change of variable  $y\sqrt{\epsilon} = x$  gives

$$\frac{1}{\sqrt{\epsilon}} \int_0^t f \left( \frac{X_u}{\sqrt{\epsilon}} \right) du = \int_{\mathbb{R}} f(y) L_t^{y\sqrt{\epsilon}}(X) dy.$$

Then, the Hölder property of the Brownian local time implies that

$$\left| \int_{\mathbb{R}} f(y) L_t^{y\sqrt{\epsilon}}(X) dy - \int_{\mathbb{R}} f(y) dy L_t^0(X) \right| \leq C \int_0^\infty (\sqrt{\epsilon} y)^\delta f(y) dy \leq C\epsilon^{\frac{\delta}{2}}. \quad \square$$

3. Finally, the lemma below ensures the convergence to 0, in the a.s. sense, of  $R_\epsilon(t)$  (a proof can be found in Section 1.3 of [2]).

**Lemma 2.3.** *Let  $x$  be a continuous function and let  $a, b : [0, 1] \times [0, T + 1] \rightarrow \mathbb{R}$ ,  $c, d : [0, 1] \times [0, T + 1]^2 \rightarrow \mathbb{R}$ , be Borel functions such that*

$$0 \leq b(\epsilon, t) - a(\epsilon, t) \leq \epsilon, \quad |c(\epsilon, s, t) - d(\epsilon, s, t)| \leq \epsilon, \quad t, s \in [0, T + 1], \epsilon \in [0, 1].$$

*Let us consider  $(A_s)_{s \in [0, T+1]}$  a collection of measurable events and*

$$R_\epsilon(t) = \frac{1}{\epsilon} \int_{a(\epsilon, t)}^{b(\epsilon, t)} (x_{c(\epsilon, s, t)} - x_{d(\epsilon, s, t)}) \mathbb{1}_{A_s} ds, \quad 0 \leq t \leq T. \quad (2.2)$$

*Then,  $R_\epsilon(t)$  tends to 0 as  $\epsilon \rightarrow 0$ , uniformly on  $[0, T]$ . Furthermore, if  $x$  is the standard Brownian motion  $X$ , then a.s.  $\sup_{t \in [0, T]} |R_\epsilon(t)| \leq C\epsilon^\delta$ .*

### 3. Proof of Theorem 1.2

We begin with the study of  $I_\epsilon^3(t)$  in point 1. below, and we will deduce the convergence of  $I_\epsilon^4(t)$  in point 2. by the time reversal property.

1. *Proof of point (i).* Reasoning by stopping allows us to suppose that  $M$ ,  $\langle M \rangle$  and the total variation of  $V$  are bounded processes. Our approach is mainly based on Tanaka's formula and Fubini's theorem.

1.a. By using Tanaka's formula, we get

$$X_{(u+\epsilon) \wedge t}^+ = X_u^+ + \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)).$$

A similar equality applies to  $X^-$  instead of  $X^+$ . Applying this to  $I_\epsilon^3(t)$  leads to

$$\begin{aligned} I_\epsilon^3(t) &= \frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dX_s \right) du \\ &\quad - \frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s \leq 0, X_u > 0\}} dX_s \right) du \\ &\quad + \frac{1}{2\epsilon} \int_0^t (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)) du. \end{aligned}$$

1.b. *Study of  $\frac{1}{2\epsilon} \int_0^t (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)) du$ .*

First, let us observe that  $L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X) = \int_u^{(u+\epsilon) \wedge t} dL_s^0(X)$ . Then, Fubini's theorem gives

$$\frac{1}{2\epsilon} \int_0^t (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)) du = \frac{1}{2} \int_0^t \frac{s \wedge \epsilon}{\epsilon} dL_s^0(X).$$

Since  $L_t^0(X) = \int_0^t dL_s^0(X)$ , we obtain

$$\left| \int_0^t (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)) du - L_t^0(X) \right| \leq \int_0^T \left| \frac{s \wedge \epsilon}{\epsilon} - 1 \right| dL_s^0(X), \quad 0 \leq t \leq T.$$

Note that  $|\frac{s \wedge \epsilon}{\epsilon} - 1|$  is bounded by 2 and tends to 0 for all  $s \in ]0, T]$  as  $\epsilon \rightarrow 0$ . Consequently, Lebesgue's theorem implies the a.s. convergence of  $\frac{1}{2\epsilon} \int_0^t (L_{(u+\epsilon) \wedge t}^0(X) - L_u^0(X)) du$  to  $\frac{1}{2} L_t^0(X)$ , uniformly on  $t \in [0, T]$ .

1.c. Study of  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dX_s \right) du$ . Obviously, this term is equal to

$$\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dV_s \right) du + \frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dM_s \right) du.$$

First, we study the integral with respect to  $dV$ . Fubini's theorem leads to

$$\left| \frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dV_s \right) du \right| \leq \int_0^T \mathbb{1}_{\{X_s > 0\}} \left| \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \mathbb{1}_{\{X_u \leq 0\}} du \right| d|V|_s.$$

Let  $s \in [0, T]$  so that  $X_s > 0$ . Since  $t \rightarrow X_t$  is continuous,  $\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \mathbb{1}_{\{X_u \leq 0\}} du$  vanishes as soon as  $\epsilon$  is small enough. Observing that  $\left| \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \mathbb{1}_{\{X_u \leq 0\}} du \right|$  is bounded by 1 allows us to apply Lebesgue's convergence theorem.

Consequently,  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dV_s \right) du$  converges to 0, a.s., uniformly with respect to  $t \in [0, T]$ .

Next, we claim that point (1) of Lemma 2.1 applies with  $Y = M$  and  $H_\epsilon(s, u) = \frac{1}{\epsilon} \mathbb{1}_{\{X_s > 0, X_u \leq 0, u \leq s\}}$ . Therefore it will prove that  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s > 0, X_u \leq 0\}} dM_s \right) du$  tends to 0 in the ucp sense. Indeed, we are in the case of conditions (1) and (2.1) can be proved via Lebesgue's convergence theorem as in the proof involving  $dV_s$ .

1.d. Study of  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s \leq 0, X_u > 0\}} dX_s \right) du$ . Although this term is quite similar to the one in point 1.c, the difference between large and strict inequalities forces us to study it independently. Like for point 1.c, we use the decomposition  $dX = dM + dV$ .

As for  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s \leq 0, X_u > 0\}} dV_s \right) du$ , an approach similar to the one in 1.c above shows that this term goes to 0 a.s., uniformly with respect to  $t \in [0, T]$ .

For the integral with respect to  $dM$ , we claim that  $E(\int_0^T \mathbb{1}_{\{X_s \leq 0\}} |\tilde{g}(\epsilon, s)|^2 d\langle M \rangle_s)$  tends to 0 as  $\epsilon \rightarrow 0$ , with  $\tilde{g}(\epsilon, s) = \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \mathbb{1}_{\{X_u > 0\}} du$ . Thus, by Lemma 2.1,  $\frac{1}{\epsilon} \int_0^t \left( \int_u^{(u+\epsilon) \wedge t} \mathbb{1}_{\{X_s \leq 0, X_u > 0\}} dM_s \right) du$  will tend to 0 in the ucp sense.

To prove that  $E(\int_0^T \mathbb{1}_{\{X_s \leq 0\}} |\tilde{g}(\epsilon, s)|^2 d\langle M \rangle_s)$  tends to 0, we decompose it as a sum of two terms:

$$E \left( \int_0^T \mathbb{1}_{\{X_s < 0\}} |\tilde{g}(\epsilon, s)|^2 d\langle M \rangle_s \right) + E \left( \int_0^T \mathbb{1}_{\{X_s = 0\}} |\tilde{g}(\epsilon, s)|^2 d\langle M \rangle_s \right).$$

Like in the approach related to the  $dV$ -integral, the first term above converges to 0 as  $\epsilon \rightarrow 0$ . Since  $\tilde{g}(\epsilon, s) \leq 1$ , the second term may be bounded by  $E(\int_0^T \mathbb{1}_{\{X_s = 0\}} d\langle M \rangle_s)$ . By the occupation times formula,

$$\int_0^T \mathbb{1}_{\{X_s = 0\}} d\langle M \rangle_s = \int_{\mathbb{R}} \mathbb{1}_{\{x = 0\}} L_T^x dx = 0. \quad \square$$

2. Proof of point (ii). Let us now suppose that  $X$  is a reversible diffusion. We will use the time reversal property to reduce the convergence of  $I_\epsilon^4(t)$  to that of a term like  $I_\epsilon^3(t)$ . By the change of variable  $s = T - u - \epsilon$ ,  $I_\epsilon^4(t)$  is expressed through  $\tilde{X}$  (defined by (1.2)):

$$I_\epsilon^4(t) = \left[ \frac{1}{\epsilon} \int_{T-t}^T \tilde{X}_{(s+\epsilon) \wedge T}^- \mathbb{1}_{\{\tilde{X}_s > 0\}} ds + \frac{1}{\epsilon} \int_{T-t}^T \tilde{X}_{(s+\epsilon) \wedge T}^+ \mathbb{1}_{\{\tilde{X}_s \leq 0\}} ds \right] + R_\epsilon(t). \quad (3.1)$$

Since  $\tilde{X}$  is a semimartingale, point (i) of [Theorem 1.2](#) may be applied: the term in brackets in [\(3.1\)](#) converges to  $\frac{1}{2}(L_T^0(\tilde{X}) - L_{T-t}^0(\tilde{X}))$ .

Finally, we express  $(L_t^0(\tilde{X}))_{t \in [0, T]}$  via  $(L_t^0(X))_{t \in [0, T]}$ . Since  $x \rightarrow L_t^x(\tilde{X})$  is right-continuous, we have

$$\begin{aligned} L_T^0(\tilde{X}) - L_{T-t}^0(\tilde{X}) &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_{T-t}^T \mathbb{1}_{\{0 < \tilde{X}_u < \alpha\}} d\langle \tilde{X} \rangle_u, \\ &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \int_0^t \mathbb{1}_{\{0 < X_s < \alpha\}} \sigma^2(s, X_s) ds = L_t^0(X). \quad \square \end{aligned}$$

#### 4. Proof of [Theorem 1.4](#)

In this section,  $X$  will be a standard Brownian motion. We have to prove four distinct properties:

- (1) the (ucp) convergence of  $I_\epsilon^{4,2}(t)$  to  $\frac{1}{4}L_t^0(X)$  (see point 2.),
- (2) the (ucp) convergence of  $I_\epsilon^{3,2}(t)$  to  $\frac{1}{4}L_t^0(X)$  (see point 3.),
- (3) the rate of decay of  $I_\epsilon^{4,i}(t) - \frac{1}{4}L_t^0(X)$  as  $\epsilon \rightarrow 0$ ,  $i = 1, 2$  (see point 2.),
- (4)  $\lim_{\epsilon \rightarrow 0} I_\epsilon^{3,1}(t) = \lim_{\epsilon \rightarrow 0} I_\epsilon^{3,2}(t)$  and  $\lim_{\epsilon \rightarrow 0} I_\epsilon^{4,1}(t) = \lim_{\epsilon \rightarrow 0} I_\epsilon^{4,2}(t)$  (see point 4.).

1. First, we briefly study  $r_\epsilon^3(t)$  and  $r_\epsilon^4(t)$  (defined by [\(1.8\)](#)). We have  $r_\epsilon^4(t) = r_\epsilon^{4,1}(t) + r_\epsilon^{4,2}(t)$ , where

$$r_\epsilon^{4,1}(t) = \frac{1}{\epsilon} \int_\epsilon^t X_{v-\epsilon}^+ \mathbb{1}_{\{X_v=0\}} dv, \quad r_\epsilon^{4,2}(t) = \frac{1}{\epsilon} \int_{t-\epsilon}^t X_u^+ \mathbb{1}_{\{X_t=0\}} du.$$

The occupation times formula  $\int_0^t \mathbb{1}_{\{X_u=0\}} du = \int_{\mathbb{R}} \mathbb{1}_{\{x=0\}} L_t^x(X) dx = 0$  implies that  $r_\epsilon^3(t) = r_\epsilon^{4,1}(t) = 0$ . Applying [Lemma 2.3](#), we may conclude that  $r_\epsilon^{4,2}(t)$  goes to 0 as  $\epsilon \rightarrow 0$ , uniformly on the compact sets.

2. Next, we study the convergence of  $I_\epsilon^{4,2}(t)$  (defined by [\(1.7\)](#)). Since  $\mathbb{1}_{\{X_{(u+\epsilon)\wedge t} < 0\}}$  is not  $(\mathcal{F}_u)$ -measurable, we “approximate” it by

$$E(\mathbb{1}_{\{X_{(u+\epsilon)\wedge t} < 0\}} | \mathcal{F}_u) = E(\mathbb{1}_{\{X_{(u+\epsilon)\wedge t} - X_u < -X_u\}} | X_u) = \Phi\left(-\frac{X_u}{\sqrt{\epsilon \wedge (t-u)}}\right).$$

We introduce it in  $I_\epsilon^{4,2}(t)$ . This leads us to consider this new decomposition:

$$I_\epsilon^{4,2}(t) = A_\epsilon^1(t) + \frac{1}{\epsilon} \int_0^t X_u^+ \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) du, \quad (4.1)$$

$$\begin{aligned} \text{where } A_\epsilon^1(t) &= \frac{1}{\epsilon} \int_0^{(t-\epsilon)^+} X_u^+ \left[ \mathbb{1}_{\{X_{u+\epsilon} < 0\}} - \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) \right] du \\ &\quad + \frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t X_u^+ \left[ \mathbb{1}_{\{X_t < 0\}} - \Phi\left(-\frac{X_u}{\sqrt{t-u}}\right) \right] du \\ &\quad + \frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t X_u^+ \left[ \Phi\left(-\frac{X_u}{\sqrt{t-u}}\right) - \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) \right] du. \end{aligned}$$



The main term in (4.1) is  $\frac{1}{\epsilon} \int_0^t X_u^+ \Phi(-\frac{X_u}{\sqrt{\epsilon}}) du$ . Indeed, Lemma 2.2 applies with  $f(x) = x^+ \Phi(-x)$ . Consequently,  $\frac{1}{\epsilon} \int_0^t X_u^+ \Phi(-\frac{X_u}{\sqrt{\epsilon}}) du$  converges to  $\frac{1}{4} L_t^0(X)$  a.s., uniformly on  $[0, T]$ . Moreover,

$$\left| \frac{1}{\epsilon} \int_0^t X_u^+ \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) du - \frac{1}{4} L_t^0(X) \right| \leq C \epsilon^{\frac{\delta}{2}}, \quad \forall t \in [0, T]. \quad (4.2)$$

*Convergence of  $A_\epsilon^1(t)$  to 0.* For  $t \in [(t - \epsilon)^+, t]$ , we decompose the term in brackets in the second integral as

$$\begin{aligned} & \mathbb{1}_{\{X_t < 0\}} - \Phi\left(\frac{-X_t}{\sqrt{u + \epsilon - t}}\right) + \left[ \Phi\left(\frac{-X_u}{\sqrt{\epsilon}}\right) - \Phi\left(\frac{-X_u}{\sqrt{t - u}}\right) \right] \\ & + \left[ \Phi\left(\frac{-X_t}{\sqrt{u + \epsilon - t}}\right) - \Phi\left(\frac{-X_u}{\sqrt{\epsilon}}\right) \right]. \end{aligned}$$

Then, we obtain

$$\begin{aligned} A_\epsilon^1(t) &= D_\epsilon^1(t) + D_\epsilon^2(t) + D_\epsilon^3(t) + D_\epsilon^4(t), \\ \text{where } D_\epsilon^1(t) &= \frac{1}{\epsilon} \int_0^{(t-\epsilon)^+} X_u^+ \left[ \mathbb{1}_{\{X_{u+\epsilon} < 0\}} - \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) \right] du, \\ D_\epsilon^2(t) &= \mathbb{1}_{\{X_t < 0\}} \frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t X_u^+ du, \\ D_\epsilon^3(t) &= -\frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t X_u^+ \Phi\left(\frac{-X_t}{\sqrt{u + \epsilon - t}}\right) du, \\ D_\epsilon^4(t) &= \frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t X_u^+ \left[ \Phi\left(\frac{-X_t}{\sqrt{u + \epsilon - t}}\right) - \Phi\left(\frac{-X_u}{\sqrt{\epsilon}}\right) \right] du. \end{aligned}$$

For  $u$  fixed, Itô's formula applied to  $\phi(x, s) = \Phi\left(-\frac{x}{\sqrt{u + \epsilon - s}}\right)$  leads to

$$\phi(X_v, v) - \phi(X_u, u) = - \int_u^v \frac{\exp\left(-\frac{X_s^2}{2(u + \epsilon - s)}\right)}{\sqrt{2\pi(u + \epsilon - s)}} dX_s, \quad \forall v \in [u, u + \epsilon].$$

Since the Lebesgue measure of  $\{u; X_{u+\epsilon} = 0\}$  vanishes, this formula (used for  $v = u + \epsilon$  if  $u \in [0, (t - \epsilon)^+]$ , and for  $v = t$  if  $u \in ](t - \epsilon)^+, t]$ ) permits us to express the terms in brackets in  $D_\epsilon^1(t)$  and  $D_\epsilon^4(t)$  as stochastic integrals. Then, bringing together  $D_\epsilon^1(t)$  and  $D_\epsilon^4(t)$  gives

$$\begin{aligned} A_\epsilon^1(t) &= D_\epsilon^0(t) + D_\epsilon^2(t) + D_\epsilon^3(t), \\ \text{where } D_\epsilon^0(t) &= -\frac{1}{\epsilon} \int_0^t X_u^+ \left( \int_u^{(u+\epsilon) \wedge t} \frac{\exp\left(-\frac{X_s^2}{2(u + \epsilon - s)}\right)}{\sqrt{2\pi(u + \epsilon - s)}} dX_s \right) du. \end{aligned} \quad (4.3)$$

*Convergence of  $D_\epsilon^2(t)$  to 0.* By Lemma 2.3,  $D_\epsilon^2(t)$  tends a.s. to 0, uniformly on  $[0, T]$  and we have

$$\sup_{t \in [0, T]} |D_\epsilon^2(t)| \leq |C_\delta| \epsilon^\delta. \quad (4.4)$$

Convergence of  $D_\epsilon^3(t)$  to 0. The Hölder property of  $t \rightarrow X_t$  and some direct calculations yield

$$\sup_{t \in [0, T]} |D_\epsilon^3(t)| \leq |C_\delta| \epsilon^\delta + C \sqrt{\epsilon}. \quad (4.5)$$

Thus,  $D_\epsilon^3(t)$  converges a.s. to 0 uniformly on  $[0, T]$ .

Convergence of  $D_\epsilon^0(t)$  to 0. Let  $Y = X$  and

$$H_\epsilon(s, u) = \frac{X_u^+}{\epsilon \sqrt{2\pi(u + \epsilon - s)}} \exp\left(-\frac{X_s^2}{2(u + \epsilon - s)}\right), \quad (u < s).$$

The Cauchy–Schwarz inequality and a few direct calculations of expectations give

$$\begin{aligned} E\left[H_\epsilon^2(s, u)\right] &\leq \frac{C}{\epsilon^2} \left(\frac{u}{u + \epsilon - s}\right)^{\frac{3}{4}}, \quad s \in ]0, T[, u \in ]s, s + \epsilon[, \\ E\left(\int_0^T \left[\int_{(s-\epsilon)^+}^s H_\epsilon(s, u) du\right]^2 ds\right) &\leq C \sqrt{T\epsilon}. \end{aligned}$$

According to Lemma 2.1,  $D_\epsilon^0(t)$  converges to 0 in  $L^2(\Omega)$ , uniformly on  $[0, T]$ , and

$$E\left[\sup_{t \in [0, T]} \left(D_\epsilon^0(t)\right)^2\right] \leq C \sqrt{\epsilon}. \quad (4.6)$$

We are now able to prove that  $I_\epsilon^{4,2}(t)$  converges. It is clear that (4.1) and (4.3) imply

$$\begin{aligned} I_\epsilon^{4,2}(t) - \frac{1}{4} L_t^0(X) &= D_\epsilon^0(t) + D_\epsilon^2(t) + D_\epsilon^3(t) \\ &\quad + \left(\frac{1}{\epsilon} \int_0^t X_u^+ \Phi\left(-\frac{X_u}{\sqrt{\epsilon}}\right) du - \frac{1}{4} L_t^0(X)\right). \end{aligned} \quad (4.7)$$

The (ucp) convergence of  $I_\epsilon^{4,2}(t)$  to  $\frac{1}{4} L_t^0(X)$  as  $\epsilon \rightarrow 0$  and (1.9) are direct consequences of inequalities (4.4)–(4.6) and (4.2).

3. Next, we study the convergence of  $I_\epsilon^{3,2}(t)$  (defined by (1.6)), by the same method as in point 2. Since  $X_{u+\epsilon}^+$  is not  $(\mathcal{F}_u)$ -measurable, we “replace” it by

$$E(X_{u+\epsilon}^+ | \mathcal{F}_u) = \sqrt{\epsilon} g\left(\frac{X_u}{\sqrt{\epsilon}}\right),$$

where  $g(x) = E((G + x)^+)$  and  $G$  is a Gaussian random variable with  $\mathcal{N}(0, 1)$ -law. This leads us to consider the following decomposition of  $I_\epsilon^{3,2}(t)$ :

$$I_\epsilon^{3,2}(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t g\left(\frac{X_u}{\sqrt{\epsilon}}\right) \mathbb{1}_{\{X_u < 0\}} du + D_\epsilon^5(t) + R_\epsilon(t),$$

$$\begin{aligned} \text{where } D_\epsilon^5(t) &= \frac{1}{\epsilon} \int_0^{(t-\epsilon)^+} (X_{u+\epsilon}^+ - E(X_{u+\epsilon}^+ | \mathcal{F}_u)) \mathbb{1}_{\{X_u < 0\}} du \\ &\quad + \frac{1}{\epsilon} \int_{(t-\epsilon)^+}^t (E(X_{u+\epsilon}^+ | \mathcal{F}_t) - E(X_{u+\epsilon}^+ | \mathcal{F}_u)) \mathbb{1}_{\{X_u < 0\}} du. \end{aligned}$$

$\frac{1}{\sqrt{\epsilon}} \int_0^t g\left(\frac{X_u}{\sqrt{\epsilon}}\right) \mathbb{1}_{\{X_u < 0\}} du$  is the main term. Since  $\int_{-\infty}^0 g(y) dy = \frac{1}{4}$ , Lemma 2.2 gives  $|\tilde{I}_\epsilon^{3,2}(t) - \frac{1}{4} L_t^0(X)| \leq C\epsilon^{\frac{\delta}{2}}$ . Therefore,  $\tilde{I}_\epsilon^{3,2}(t)$  converges to  $\frac{1}{4} L_t^0(X)$  a.s., uniformly on  $[0, T]$ , as soon as  $D_\epsilon^5(t)$  goes to 0 (see the proof below).

*Study of  $D_\epsilon^5(t)$ .*

First, we write  $(X_{u+\epsilon}^+ - E(X_{u+\epsilon}^+ | \mathcal{F}_u))$  and  $(E(X_{u+\epsilon}^+ | \mathcal{F}_t) - E(X_{u+\epsilon}^+ | \mathcal{F}_u))$  as stochastic integrals, by using an approach similar to the one considered in the convergence of  $A_\epsilon^1(t)$  (see point 2 above). We obtain

$$D_\epsilon^5(t) = \int_0^t \left[ \int_u^{(u+\epsilon) \wedge t} \frac{1}{\epsilon} \left\{ 1 - \Phi\left(\frac{-X_s}{\sqrt{u+\epsilon-s}}\right) \right\} dX_s \right] \mathbb{1}_{\{X_u < 0\}} du.$$

Since  $x \rightarrow 1 - \Phi(x)$  is bounded by 1, condition (2) of Lemma 2.1 holds. Next, we show that

$$\lim_{\epsilon \rightarrow 0} E \left( \int_0^T \left[ \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \left\{ 1 - \Phi\left(\frac{-X_s}{\sqrt{u+\epsilon-s}}\right) \right\} \mathbb{1}_{\{X_u < 0\}} du \right]^2 ds \right) = 0. \quad (4.8)$$

Since the Lebesgue measure of  $\{s : X_s = 0\}$  is null, we can suppose that either  $X_s > 0$  or  $X_s < 0$ .

- (1) For  $s$  such that  $X_s > 0$ , the term in brackets is bounded by  $\mathbb{1}_{\{X_s > 0\}} \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^s \mathbb{1}_{\{X_u < 0\}} du$ . We have already proven in point 1.c that this term goes to 0, as  $\epsilon \rightarrow 0$ , a.s. and in  $L^1(\Omega)$ .
- (2) Let us assume  $X_s < 0$ . Using  $|1 - \Phi(\alpha)| \leq C e^{-\frac{\alpha^2}{4}}$  for any  $\alpha > 0$ , and the change of variable  $v = u + \epsilon - s$ , it can be shown that the term in brackets in (4.8) is bounded by  $\frac{C}{\epsilon} \int_{(\epsilon-s)^+}^\epsilon e^{-\frac{X_s^2}{4v}} dv$ . It is easy to deduce that it converges to 0 a.s. and in  $L^1(\Omega)$ , as  $\epsilon \rightarrow 0$ .

Then, (4.8) is a direct consequence of Lebesgue's theorem. Finally, Lemma 2.1 permits us to conclude that  $D_\epsilon^5(t)$  goes to 0, in the ucp sense.

4. Let us observe that  $L_t^0(X) = L_t^0(-X)$  and the transformation  $X \rightarrow -X$  leads to  $I_\epsilon^{4,1}(t) \rightarrow I_\epsilon^{4,2}(t)$  and  $I_\epsilon^{3,1}(t) \rightarrow I_\epsilon^{3,2}(t)$ . Consequently, Theorem 1.4 follows from the above steps.  $\square$

## 5. Proofs of Theorem 1.5 and Proposition 1.6

1. In this section,  $X$  is supposed to be the standard Brownian motion. By Tanaka's formula, we decompose  $J_\epsilon(t) - L_t^0(X)$  as

$$J_\epsilon(t) - L_t^0(X) = - \left( I_\epsilon^1(t) - \int_0^t \mathbb{1}_{\{0 < X_s\}} dX_s \right) + \left( I_\epsilon^2(t) - X_t^+ - \frac{1}{2} L_t^0 \right) + R_\epsilon(t). \quad (5.1)$$

$$\text{where } I_\epsilon^1(t) = \int_0^t \frac{X_{(s+\epsilon) \wedge t} - X_s}{\epsilon} \mathbb{1}_{\{0 < X_s\}} ds,$$

$$I_\epsilon^2(t) = \int_0^t \frac{X_{(s+\epsilon) \wedge t} - X_s}{\epsilon} \mathbb{1}_{\{0 < X_{(s+\epsilon) \wedge t}\}} ds.$$

Therefore, Theorem 1.5 comes from Lemma 2.3 and the convergence of  $I_\epsilon^1(t)$  (step 2) and  $I_\epsilon^2(t)$  (step 3). In step 4 we will show Proposition 1.6.

2. *Convergence of  $I_\epsilon^1(t)$ .* Writing  $X_{(s+\epsilon)\wedge t} - X_s$  as  $\int_s^{(s+\epsilon)\wedge t} dX_u$  and using Fubini's stochastic theorem gives

$$I_\epsilon^1(t) - \int_0^t \mathbb{1}_{\{0 < X_s\}} dX_s = \int_0^t \left( \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u \mathbb{1}_{\{0 < X_s\}} ds - \mathbb{1}_{\{0 < X_u\}} \right) dX_u.$$

We can prove (see Section 3.4.2 of [2] for the details of this fastidious calculation) that

$$E \left[ \int_0^T \left( \frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u \mathbb{1}_{\{0 < X_s\}} - \mathbb{1}_{\{0 < X_u\}} ds \right)^2 du \right] \leq C \sqrt{\epsilon}.$$

Then, Doob's inequality gives

$$E \left( \sup_{0 \leq t \leq T} \left( I_\epsilon^1(t) - \int_0^t \mathbb{1}_{\{0 < X_s\}} dX_s \right)^2 \right) \leq C \sqrt{\epsilon}. \quad (5.2)$$

3. *Study of the convergence of  $I_\epsilon^2(t)$ .* The idea of our approach is to express  $I_\epsilon^2(t)$  through  $I_\epsilon^{4,1}(t)$  and  $I_\epsilon^{4,2}(t)$ . Namely, we have

$$I_\epsilon^2(t) - X_t^+ - \frac{1}{2} L_t^0(X) = \left( I_\epsilon^{4,1}(t) - \frac{1}{4} L_t^0(X) \right) + \left( I_\epsilon^{4,2}(t) - \frac{1}{4} L_t^0(X) \right) + R_\epsilon(t) + r_\epsilon^4(t),$$

(recall that  $I_\epsilon^{4,1}(t)$ ,  $I_\epsilon^{4,2}(t)$ ,  $r_\epsilon^4(t)$  are defined by (1.7) and (1.8)).

It is now clear that (1.10) follows from Lemma 2.3 and Theorem 1.4.

4. We now demonstrate Proposition 1.6. Let us consider a sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  of positive real numbers decreasing to 0, such that  $\sum_{i=1}^\infty \sqrt{\epsilon_i} < \infty$ . Note that Theorems 1.4 and 1.5 do not permit us to obtain the a.s. convergence results stated in Proposition 1.6, via the Borel–Cantelli lemma, since we may not take  $\delta = \frac{1}{2}$ .

Identity (4.6) implies that the Borel–Cantelli Lemma may be applied:  $(D_{\epsilon_n}^0(t))_{n \in \mathbb{N}}$  converge almost surely uniformly for  $t \in [0, T]$ . Consequently, the a.s. convergence of  $I_{\epsilon_n}^{4,2}(t)$  to  $\frac{1}{4} L_t^0(X)$  as  $n \rightarrow \infty$  is a direct consequence of (4.7), (4.2), (4.4) and (4.5).

The convergence of  $I_{\epsilon_n}^{4,1}(t)$  (resp.  $J_{\epsilon_n}(t)$ ) to  $\frac{1}{4} L_t^0(X)$  (resp.  $\frac{1}{2} L_t^0(X)$ ) may be obtained similarly.  $\square$

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